ON HOMOGENEOUS PINNING MODELS AND PENALIZATIONS

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Abstract. In this note, we show how the penalization method, introduced in order to describe some non-trivial changes of the Wiener measure, can be applied to the study of some simple polymer models such as the pinning model. The bulk of the analysis is then focused on the study of a martingale which has to be computed as a Markovian limit.

1. Introduction

Our motivation for writing the current note is the following: on the one hand, in the last past years, some interesting advances have seen the light concerning various kind of polymer models, having either an interaction with a random environment or a kind of intrinsic self-interaction. Among this wide class of models, we will be interested here in some polymers interacting with a given interface, as developed for instance in [1, 7]. For this kind of polymers, the introduction of some generalized renewal tools has yield some very substantial progresses in the analysis of the model, and a quite complete picture of their asymptotic behaviour in terms of localization near the interface is now available e.g. in [5, 6] and in the monograph [4].

On the other hand, and a priori in a different context, the series of papers starting by [8] and ending with the recent monograph [9] presents a rather simple method in order to quantify the penalization of a Brownian (or Bessel) path by a functional of its trajectory (such as the one-sided supremum or the number of excursions). This method can then be applied in a wide number of natural situations, getting a very complete description of some Gibbs type measures based on the original Brownian motion. More specifically, when translated in a random walk context, the penalization method can be read as follows: let $\{b_n;\, n \ge 0\}$ be a symmetric random walk on $\mathbb{Z}$, defined on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\ge1}, (\mathbb{P}_z)_{z\in\mathbb{Z}})$. For $n \ge 0$, let also e^{H_n} be a bounded positive measurable functional of the path $(b_0, \ldots, b_n)$. Then, for $\beta \in \mathbb{R}$, $n \ge p \ge 0$, we are concerned with a generic Gibbs type measure ρ_n on $\mathcal{F}_p$ defined, for $\Gamma_p \in \mathcal{F}_p$, by

$$\rho_n(\Gamma_p) = \frac{\mathbb{E}_0\left[\mathbb{1}_{\Gamma_p} e^{\beta H_n}\right]}{Z_n}, \quad \text{where} \quad Z_n = \mathbb{E}_0\left[e^{\beta H_n}\right]. \tag{1.1}$$

In its general formulation, the penalization principle, which allows an asymptotic study of ρ_n, can be stated as follows:

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Proposition 1.1. *Suppose that the process (b_n, H_n) is a $\mathbb{Z}\times\mathbb{R}_+$-valued Markov process, and let Λ_n be its semi-group. Assume that, for any $p \ge 0$, the function M_p defined by*

$$M_p(w,z) := \lim_{n\to\infty} \frac{[\Lambda_{n-p} f](w,z)}{[\Lambda_n f](0)}, \qquad \textit{where} \qquad f(w,z) = e^{-\beta z} \tag{1.2}$$

exists, for any $(w,z) \in \mathbb{Z}\times\mathbb{R}_+$, and that

$$\frac{[\Lambda_{n-p} f](w,z)}{[\Lambda_n f](0)} \le C(p,w,z), \qquad \textit{where} \qquad \mathbb{E}_0[C(p, b_p, \ell_p)] < \infty.$$

Then:

1. *the process $M_p := M_p(b_p, \ell_p)$ is a non-negative $\mathbb{P}_0$-martingale;*
2. *for any $p \ge 0$, when $n \to \infty$, the measure ρ_n defined by (1.1) converges weakly on $\mathcal{F}_p$ to a measure ρ, where ρ is defined by*

$$\rho(\Gamma_p) = \mathbb{E}_0\left[\mathbb{1}_{\Gamma_p} M_p\right], \qquad \textit{for} \qquad \Gamma_p \in \mathcal{F}_p.$$

This last proposition can be seen then as an invitation to organize the asymptotic study of the measure ρ_n in the following way: first compute explicitly the limit of the ratio $[\Lambda_{p-n} f](w,z)/[\Lambda_p f](0)$ when $p \to \infty$, which should define also an asymptotic measure ρ in the infinite volume regime. Then try to read the basic properties of ρ by taking advantage of some simple relations on the martingale M_p.

It is easily seen that some links exists between the polymer measure theory as mentioned above and the penalization method. Furthermore, we believe that the two theories can interact in a fruitful way. Indeed, the penalizing scheme offers a simple and systematic framework for the study of Gibbs measures based on paths, and it is also quite pleasant to be able to read the main features of the limiting measure ρ on the martingale M_p, which is usually a simple object. Without presenting a completely new result, this article will thus try to make a bridge between the two aspects of the topic, by studying the simplest of the interface-based polymers, namely the polymer pinned at an interface, through a purely penalizing scheme. Let us be more specific once again, and describe our model and the main results we shall obtain: denote by ℓ_n the local time at 0 of b, that is

$$\ell_n = \sharp\{p \le n;\, b_p = 0\}.$$

For $\beta \in \mathbb{R}$, $n \ge p \ge 0$, we are concerned here with the Gibbs type measure $\mathbb{Q}_0^{(n,\beta)}$ on $\mathcal{F}_p$ defined, for $\Gamma_p \in \mathcal{F}_p$, $p < n$, by

$$\mathbb{Q}_0^{(n,\beta)}(\Gamma_p) = \frac{\mathbb{E}_0\left[\mathbb{1}_{\Gamma_p} e^{\beta \ell_n}\right]}{Z_n^f}, \qquad \text{where} \qquad Z_n^f = \mathbb{E}_0\left[e^{\beta\ell_n}\right]. \tag{1.3}$$

Finally, we will need to introduce a slight variation of the Bessel walk of dimension 3, which is defined as a random walk R on $\mathbb{N}$ starting from 0, such that $\mathbb{P}_0(R_0 = 0) = \mathbb{P}_0(R_1 = 1) = 1$, and whenever $j \ge 1$,

$$\mathbb{P}_0(R_{n+1} = j \pm 1 \,|\, R_n = j) = \frac{j \pm 1}{2j}. \tag{1.4}$$

With these notations in hand, the main result we shall obtain is then the following:

Theorem 1.1. *For $\beta \in \mathbb{R}$, $n \geq p \geq 0$, let $\mathbb{Q}_0^{(n,\beta)}$ be the measure defined by (1.3). Then, for any $p \geq 0$, the measure $\mathbb{Q}_0^{(n,\beta)}$ on $\mathcal{F}_p$ converges weakly, as $n \to \infty$, to a measure $\mathbb{Q}_0^{(\beta)}$ defined by*

$$\mathbb{Q}_0^{(\beta)}(\Gamma_p) = \mathbb{E}_0\left[\mathbb{1}_{\Gamma_p} M_p^{(\beta)}\right], \quad \text{for} \quad \Gamma_p \in \mathcal{F}_p. \tag{1.5}$$

According to the sign of β the two following situations can occur:

(1) *When $\beta < 0$ (delocalized phase): set $\alpha = -\beta$. Then $M_p^{(\beta)}$ has the following expression:*

$$M_p^{(\beta)} = e^{-\alpha \ell_p}\left[(1 - e^{-\alpha})|b_p| + 1\right].$$

Moreover, under the probability $\mathbb{Q}_0^{(\beta)}$, the process b and its local time ℓ can be described in the following way:

a) *The random variable ℓ_∞ is finite almost surely, and is distributed according to a geometric law with parameter $1 - e^{-\alpha}$.*
b) *Let $g = \sup\{r \geq 0;\, b_r = 0\}$. Then g is finite almost surely, and the two processes $b^{(-)} = \{b_r;\, r \leq g\}$ and $b^{(+)} = \{b_{r+g};\, r \geq 0\}$ are independent.*
c) *The process $|b^{(+)}|$ is a Bessel random walk as defined by the transition law (1.4), and sign$(b^{(+)}) = \pm 1$ with probability $1/2$.*
d) *Given the event $\ell_\infty = l$ for $l \geq 1$, the process $b^{(-)}$ is a standard random walk, stopped when its local time reaches l.*

(2) *When $\beta > 0$ (localized phase): in this case, the martingale $M_p^{(\beta)}$ can be written as:*

$$M_p^{(\beta)} = \exp\left\{\beta \hat{\ell}_p - c_{+,\beta}\,|b_p| - c_{-,\beta}\, p\right\}, \tag{1.6}$$

where $c_{\pm,\beta} = (1/2)[\beta \pm \ln(2 - e^{-\beta})]$, and where $\hat{\ell}_p$ is a slight modification of ℓ_p defined by $\hat{\ell}_p = \ell_p - \mathbb{1}_{b_p=0}$. Furthermore, under the probability $\mathbb{Q}_0^{(\beta)}$, the process b can be decomposed as follows:

a) *Let $\tau = (\tau_0^j)_{j\geq 1}$ be the successive return times of b at 0, and set $\tau_0^0 = 0$, $\tau_0^1 = \tau_0$. Then the sequence $\{\tau_0^j - \tau_0^{j-1};\, j \geq 1\}$ is i.i.d, and the law of τ_0 is defined by its Laplace transform (5.4). In particular, τ_0 has a finite mean, whose equivalent, as $\beta \to \infty$, is $1 - e^{-\beta}/2$.*
b) *Given the sequence τ, the excursions $(b^j)_{j\geq 1}$, defined by $b_r^j = b_{\tau_0^{j-1}+r}$ for $r \leq \tau_0^j - \tau_0^{j-1}$, are independent. Moreover, each b^j is distributed as a random walk starting from 0, constrained to go back to 0 at time $\tau_0^j - \tau_0^{j-1}$.*

As mentioned above, the results presented in this note are not really new. In the penalization literature, the random walk weighted by a functional of its local time has been considered by Debs in [2] for the delocalized phase, and we only cite his result here in order to give a complete picture of our polymer behaviour. We shall thus concentrate on the localized phase $\beta > 0$ in the remainder of the article. However, in this case the

results concerning homogeneous polymers can be considered now as classical, and the first rigorous treatment of our pinned model can be traced back at least to [1]. The results we obtain for the localized part of our theorem can also be found, in an (almost) explicit way, in [5, 4]. But once again, our goal here is just to show that the penalization method can be applied in this context, and may shed a new light on the polymer problem. Furthermore, we believe that this method may be applied to other continuous or discrete inhomogeneous models, hopefully leading to some simplifications in their analysis. These aspects will be handled in a subsequent publication.

Let us say now a few words about the way our article is structured: at Section 2, we will recall some basic identities in law for the simple symmetric random walk on $\mathbb{Z}$. In order to apply our penalization program, a fundamental step is then to get some sharp asymptotics for the semi-group Λ_n mentioned at Proposition 1.1. This will be done at Section 3, thanks to the renewal trick introduced e.g. in [4]. This will allow to us to describe our infinite volume limit at Section 4 in terms of the martingale $M_p^{(\beta)}$. The description of the process b under the infinite volume measure given at Theorem 1.1 will then be proved, in terms of the behavior of $M_p^{(\beta)}$, at Section 5.

2. Classical facts on random walks

Let us first recall some basic results about the random walk b: for $n \geq 0$ and $z \in \mathbb{Z}$, set

$$S_n = \sup\{b_p;\, p \leq n\}, \quad T_z = \inf\{n \geq 0;\, b_n = z\} \quad \text{and} \quad \tau_z = \inf\{n \geq 1;\, b_n = z\}.$$

Let us denote by $\mathbb{D}$ the set of even integers in $\mathbb{Z}$, and for $(n,r) \in \mathbb{N} \times \mathbb{Z}$, recall that $p_{n,r} := \mathbb{P}_0(b_n = r)$ is given by:

$$p_{n,r} = \left(\frac{1}{2}\right)^n \binom{(n+r)/2}{n} \mathbb{1}_{\mathbb{D}}(n+r)\mathbb{1}_{\{|r| \leq n\}}.$$

Then it is well-known (see e.g. [3, 2]) that

$$\mathbb{P}_0(S_n = r) = p_{n,r} \vee p_{n,r+1} \quad \text{and} \quad \mathbb{P}_0(T_r = n) = \frac{r}{n}\left(\frac{1}{2}\right)^n \binom{(n+r)/2}{n}. \tag{2.1}$$

Moreover, the distribution of ℓ_n can be expressed in terms of these quantities:

$$\mathbb{P}_0(\ell_n = k) = \mathbb{P}_0(S_{n-k} = k) + \mathbb{P}_0(T_{n+1} = n-k), \tag{2.2}$$

and the following asymptotic results hold true:

Lemma 2.1. *Let $p \in \mathbb{N}$ and set $\kappa = (2/\pi)^{1/2}$. Then*

$$\lim_{n\to\infty} n^{1/2}\mathbb{P}_0(S_n = p) = \lim_{n\to\infty} n^{1/2}\mathbb{P}_0(\ell_n = p) = \kappa, \quad \text{and} \quad \lim_{n\to\infty} n^{3/2}\mathbb{P}_0(T_z = n) = \kappa z.$$

For our further computations, we will also need the following expression for the Laplace transform of T_r and τ_r:

Lemma 2.2. *Let $r \in \mathbb{N}$, $\delta > 0$. Then*

$$\mathbb{E}_0[e^{-\delta T_r}] = \exp\left\{-r \arg\cosh(e^\delta)\right\} \tag{2.3}$$

and

$$\mathbb{E}_0[e^{-\delta \tau_r}] = \begin{cases} \exp\left\{-r \arg\cosh(e^\delta)\right\}, & \text{if } r \geq 1 \\ \exp\left\{-\delta - \arg\cosh(e^\delta)\right\}, & \text{if } r = 0 \end{cases} \tag{2.4}$$

Proof. This is an elementary computation based on the fact that $\{\exp(\eta b_n - \delta n);\, n \geq 1\}$ is a martingale. Also, note that τ_0 has the same law as $1 + T_1$. $\square$

3. Laplace transform of the local time

Our aim in this section is to find an asymptotic equivalent for the Laplace transform Z_n^f of ℓ_n. However, for computational purposes, we will also have to consider the following constrained Laplace transform :

$$Z_{2m}^c := \mathbb{E}_0\left[e^{\beta \ell_{2m}} \mathbb{1}_{\{b_{2m}=0\}}\right] \quad (\beta \geq 0).$$

With this notation in hand, here is our first result about the exponential moments of the local time:

Lemma 3.1. *For any $\beta > 0$, we have*

$$\lim_{m\to\infty} \left(e^{-\beta}(2 - e^{-\beta})\right)^m Z_{2m}^c = c_\beta^c, \quad \text{where} \quad c_\beta^c := \frac{2(1 - e^{-\beta})}{2 - e^{-\beta}}, \tag{3.1}$$

and

$$\lim_{n\to\infty} \left(e^{-\beta}(2 - e^{-\beta})\right)^{\lfloor n/2 \rfloor} Z_n^f = c_\beta^f, \quad \text{where} \quad c_\beta^f := \frac{2}{2 - e^{-\beta}}. \tag{3.2}$$

Proof. According to (1.9)-(1.10) in [4, p. 9], by using the renewal theorem, we can write

$$Z_{2m}^c = \mathbb{E}_0\left[e^{\beta \ell_{2m}} \mathbb{1}_{\{b_{2m}=0\}}\right] = \sum_{k=1}^m \sum_{\mathbf{r}\in A_{k,m}} \prod_{j=1}^k e^\beta \mathbb{P}_0(\tau_0 = 2r_j) \underset{m\to\infty}{\sim} \frac{e^{m\mathtt{F}(\beta)}}{\sum_m m\tilde{K}_\beta(m)}, \tag{3.3}$$

where we denoted $A_{k,m} = \{\mathbf{r} = (r_1, \ldots, r_k), \sum_{j=1}^k r_j = m\}$. Here

$$\tilde{K}_\beta(m) := \exp\left(\beta - m\mathtt{F}(\beta)\right) K(m), \quad \text{where} \quad K(m) := \mathbb{P}_0(\tau_0 = 2m), \tag{3.4}$$

and $\mathtt{F}(\beta)$ is the solution of the following equation (see also (1.6), p. 8 in [4])

$$\sum_m e^{-m\mathtt{F}(\beta)} K(m) = e^{-\beta} \quad \text{i.e.} \quad \mathbb{E}_0\left[e^{-\mathtt{F}(\beta)\tau_0/2}\right] = e^{-\beta}. \tag{3.5}$$

Notice that in our case, equation (3.5) can be solved explicitly: thanks to relation (2.4), it can be transformed into:

$$\exp\left(-\mathtt{F}(\beta)/2 - \arg\cosh\left(e^{\mathtt{F}(\beta)/2}\right)\right) = e^{-\beta} \Leftrightarrow \cosh\left(\beta - \mathtt{F}(\beta)/2\right) = e^{\mathtt{F}(\beta)/2} \Leftrightarrow e^{\beta - \mathtt{F}(\beta)} + e^{-\beta} = 2,$$

and thus, the solution of (3.5) is given by

$$\mathtt{F}(\beta) = \beta - \ln(2 - e^{-\beta}). \tag{3.6}$$

On the other hand,

$$\sum_m me^{-\lambda m}\mathbb{P}_0(\tau_0=2m)=-\frac{d}{d\lambda}\mathbb{E}_0\left[e^{-\lambda\tau_0/2}\right]=-\frac{d}{d\lambda}\left(1-e^{-\lambda/2}(e^{\lambda}-1)^{1/2}\right)=\frac{e^{-\lambda}}{2(1-e^{-\lambda})^{1/2}},$$

as we can see again by (2.4) and simple computation. Therefore, taking $\lambda=\mathtt{F}(\beta)$, we obtain

$$\sum_m m\tilde{K}_\beta(m)=e^{\beta}\sum_m me^{-m\mathtt{F}(\beta)}\mathbb{P}_0(\tau_0=2m)=\frac{2-e^{-\beta}}{2(1-e^{-\beta})},\tag{3.7}$$

since, according to (3.6), $e^{-\mathtt{F}(\beta)}=e^{-\beta}(2-e^{-\beta})=1-(1-e^{-\beta})^2$. Puting together (3.3), (3.6) and (3.7) we get the equivalent for the constrained Laplace transform (3.1).

We proceed now with the study of the free Laplace transform, called Z_n^f. Set $\overline{K}(n):=\sum_{j>n}K(j)$. We can write

$$\begin{aligned}Z_{2m}^f&=\sum_{j=0}^m\mathbb{E}_0\left[e^{\beta\ell_{2m}}\mathbb{1}_{\max\{k\le m,b_{2k}=0\}=j}\right]=\sum_{j=0}^m\mathbb{E}_0\left[e^{\beta\ell_{2j}}\mathbb{1}_{\{b_{2j}=0\}}\mathbb{1}_{\{\tau_0\circ\theta_{2j}>2(m-j)\}}\right]\\&=\sum_{j=0}^m\mathbb{E}_0\left[e^{\beta\ell_{2j}}\mathbb{1}_{\{b_{2j}=0\}}\right]\mathbb{P}_0\left(\tau_0>2(m-j)\right)=\sum_{j=0}^m\mathbb{E}_0\left[e^{\beta\ell_{2(m-j)}}\mathbb{1}_{\{b_{2(m-j)}=0\}}\right]\overline{K}(j)\\&=\sum_{j=0}^m Z_{2(m-j)}^c\overline{K}(j)=e^{m\mathtt{F}(\beta)}\sum_{j=0}^m e^{-(m-j)\mathtt{F}(\beta)}\,Z_{2(m-j)}^c e^{-j\mathtt{F}(\beta)}\overline{K}(j).\end{aligned}$$

In order to use (3.1) on the right hand side of the latter equality we need to apply the dominated convergence theorem. This is allowed by the inequality

$$e^{-(m-j)\mathtt{F}(\beta)}\,Z_{2(m-j)}^c\le 1,\tag{3.8}$$

which is valid since $e^{-j\mathtt{F}(\beta)}Z_{2j}^c$ represents the probability that a random walk with positive increments with law $\tilde{K}_\beta$ passes by j (see also (1.9) in [4], p. 9). Therefore, according to (3.1) and (2.4),

$$\begin{aligned}Z_{2m}^f\underset{m\to\infty}{\sim}c_\beta^c e^{m\mathtt{F}(\beta)}\sum_{j=0}^\infty e^{-j\mathtt{F}(\beta)}\sum_{i=j+1}^\infty K(i)&=c_\beta^c e^{m\mathtt{F}(\beta)}\sum_{i=1}^\infty K(i)\sum_{j=0}^{i-1}e^{-j\mathtt{F}(\beta)}\\&=\frac{c_\beta^c}{1-e^{-\mathtt{F}(\beta)}}e^{m\mathtt{F}(\beta)}\left(\sum_{i=1}^\infty K(i)-\sum_{i=1}^\infty K(i)e^{-i\mathtt{F}(\beta)}\right)=\frac{c_\beta^c e^{m\mathtt{F}(\beta)}(1-e^{-\beta})}{1-e^{-\mathtt{F}(\beta)}}=\frac{2e^{m\mathtt{F}(\beta)}}{2-e^{-\beta}}\end{aligned}$$

and we get (3.2), by using (3.6). To finish the proof, let us note that, for any $\beta>0$,

$$\mathbb{E}_0\left[e^{\beta\ell_{2m+1}}\right]=\mathbb{E}_0\left[e^{\beta\ell_{2m}}\right]=Z_{2m}^f\underset{m\to\infty}{\sim}c_\beta^f e^{m\mathtt{F}(\beta)}.\tag{3.9}$$

□

We will now go one step further and give an equivalent of $\mathbb{E}_x\left[e^{\beta\ell_n}\right]$ for an arbitrary $x\in\mathbb{Z}$. Let us denote by $\mathbb{O}$ the set of odd integers in $\mathbb{Z}$.

Lemma 3.2. *Let $x \in \mathbb{Z}$ be the starting point for b and recall that the constant c_β^f has been defined at relation (3.2). Then, for any $\beta > 0$,*

$$\mathbb{E}_x\left[e^{\beta\ell_n}\right] \underset{n\to\infty}{\sim} c_\beta^f \exp\left\{\frac{\mathtt{F}(\beta)}{2}\left(n+|x|-\mathbb{1}_{\mathbb{O}}(n+x)\right)-\beta|x|\right\}. \tag{3.10}$$

Proof. First of all, notice that, by symmetry of the random walk, $\mathbb{E}_x\left[e^{\beta\ell_n}\right] = \mathbb{E}_{-x}\left[e^{\beta\ell_n}\right]$. We will thus treat the case of a strictly positive initial condition x without loss of generality.

Case $x, n \in \mathbb{D}$. Let us split $\mathbb{E}_x\left[e^{\beta\ell_{2m}}\right]$ into

$$\mathbb{E}_x\left[e^{\beta\ell_{2m}}\right] = \mathbb{P}_x\left(T_0 > 2m\right) + \mathbb{E}_x\left[e^{\beta\ell_{2m}}\mathbb{1}_{\{T_0\le 2m\}}\right] =: D_1(2m) + D_2(2m).$$

Then, on the one hand,

$$D_1(2m) = \mathbb{P}_0\left(T_x > 2m\right) = \mathbb{P}_0\left(S_{2m} < x\right),$$

and thus, owing to Lemma 2.1, we have

$$D_1(2m) \underset{m\to\infty}{\sim} \kappa x m^{-1/2}. \tag{3.11}$$

On the other hand, setting $g(p) = \mathbb{E}_0\left[e^{\beta\ell_p}\right]$, we can write

$$\begin{aligned}
D_2(2m) &= \mathbb{E}_x\left[\mathbb{1}_{\{T_0\le 2m\}}g(2m-T_0)\right] = \sum_{k=0}^{m}\mathbb{P}_x(T_0=2k)g(2(m-k)) \\
&= e^{m\mathtt{F}(\beta)}\sum_{k=0}^{m}\mathbb{P}_x(T_0=2k)e^{-k\mathtt{F}(\beta)}g(2(m-k))e^{-(m-k)\mathtt{F}(\beta)} \\
&\underset{m\to\infty}{\sim} c_\beta^f e^{m\mathtt{F}(\beta)}\mathbb{E}_0\left[\exp\left\{-\mathtt{F}(\beta)\frac{T_x}{2}\right\}\right] = c_\beta^f \exp\left\{m\mathtt{F}(\beta) - \operatorname{arg\,cosh}\left(\frac{\mathtt{F}(\beta)}{2}\right)x\right\} \\
&= c_\beta^f \exp\left\{\frac{\mathtt{F}(\beta)}{2}(2m+x)-\beta x\right\},
\end{aligned}$$

which is (3.10). Here we used the dominated convergence theorem allowed again by the fact that $g(2(m-k))e^{-(m-k)\mathtt{F}(\beta)} \le 1$ (this inequality being obtained by a little elaboration of (3.8)).

Case $x \in \mathbb{D}, n \in \mathbb{O}$. Clearly, invoking the latter result, we have

$$\mathbb{E}_x\left[e^{\beta\ell_n}\right] = \mathbb{E}_x\left[e^{\beta\ell_{n-1}}\right] \underset{n\to\infty}{\sim} c_\beta^f \exp\left\{\frac{\mathtt{F}(\beta)}{2}(n-1+x)-\beta x\right\}.$$

Case $x \in \mathbb{O}, n \in \mathbb{D}$. Following a similar reasoning as for the first case, we see that it is enough to study the term $D_2(2m)$:

$$\begin{aligned}
D_2(2m) &= \mathbb{E}_x\left[\mathbb{1}_{\{T_0 \le 2m\}} g(2m - T_0)\right] = \sum_{k=1}^{m} \mathbb{P}_x(T_0 = 2k-1) g(2m - 2k + 1) \\
&= \sum_{k=1}^{m} \mathbb{P}_x(T_0 = 2k-1) g(2(m-k)) \underset{m\to\infty}{\sim} c_\beta^f e^{m\mathtt{F}(\beta)} \sum_{k=1}^{\infty} \mathbb{P}_x(T_0 = 2k-1) e^{-k\mathtt{F}(\beta)} \\
&= c_\beta^f e^{m\mathtt{F}(\beta)} \mathbb{E}_x\left[\exp\left\{-\mathtt{F}(\beta)\frac{1+T_0}{2}\right\}\right] = c_\beta^f e^{(m-1/2)\mathtt{F}(\beta)} \mathbb{E}_0\left[\exp\left\{-\mathtt{F}(\beta)\frac{T_x}{2}\right\}\right] \\
&= c_\beta^f e^{(m-1/2)\mathtt{F}(\beta)} \exp\left\{\left(\frac{\mathtt{F}(\beta)}{2} - \beta\right)\right\} = c_\beta^f \exp\left\{\frac{\mathtt{F}(\beta)}{2}(2m - 1 + x) - \beta x\right\}.
\end{aligned}$$

Here we used again the dominated convergence theorem and the fact that $\ell_{2(m-k)+1}$ and $\ell_{2(m-k)}$ have the same law under $\mathbb{P}_0$.

Case $x, n \in \mathbb{O}$. Again, by using the preceding result

$$\mathbb{E}_x\left[e^{\beta\ell_n}\right] = \mathbb{E}_x\left[e^{\beta\ell_{n+1}}\right] \underset{n\to\infty}{\sim} c_\beta^f \exp\left\{\frac{\mathtt{F}(\beta)}{2}(n + x) - \beta x\right\}.$$

□

4. Gibbs limit

Let us turn now to the asymptotic behaviour of the measure $\mathbb{Q}_0^{(n,\beta)}$ defined at (1.3). To this purpose, we will need an additional definition: for $n \ge 0$, let $\hat{\ell}_n$ be the modified local time given by:

$$\hat{\ell}_n = \ell_n - \mathbb{1}_{\{b_n = 0\}},$$

and notice that this modified local time appears here because ℓ satisfies the relation

$$\ell_n = \hat{\ell}_p + \ell_{n-p} \circ \theta_p \quad \text{instead of} \quad \ell_n = \ell_p + \ell_{n-p} \circ \theta_p.$$

Indeed, it is readily checked that one zero is doubly counted in the latter relation if $b_p = 0$.

With this notation in hand, the limit of $\mathbb{Q}$ is given by the following:

Proposition 4.1. *For any $p \ge 0$, the measure $\mathbb{Q}_0^{(n,\beta)}$ converges weakly on $\mathcal{F}_p$, as $n \to \infty$, to the measure $\mathbb{Q}_0^{(\beta)}$ given by*

$$\mathbb{Q}_0^{(\beta)}(\Gamma_p) = \mathbb{E}_0\left[\mathbb{1}_{\Gamma_p} M_p^{(\beta)}\right], \quad \textit{for} \quad \Gamma_p \in \mathcal{F}_p, \tag{4.1}$$

with $M^{(\beta)}$ a positive martingale defined by

$$M_p^{(\beta)} = \exp\left\{\beta\hat{\ell}_p - c_+(\beta)\,|b_p| - c_-(\beta)\,p\right\}, \tag{4.2}$$

where

$$c_\pm(\beta) = (1/2)[\beta \pm \ln(2 - e^{-\beta})]. \tag{4.3}$$

Proof. For $n \geq p$, let us decompose ℓ_n into

$$\ell_n = \hat{\ell}_p + \ell_{n-p} \circ \theta_p.$$

Thanks to this decomposition, we obtain, for a given $\Gamma_p \in \mathcal{F}_p$,

$$\mathbb{Q}_0^{(n,\beta)}(\Gamma_p) = \mathbb{E}_0\left[\mathbb{1}_{\Gamma_p} e^{\beta\hat{\ell}_p} U_{n,p}(b_p)\right], \quad \text{with} \quad U_{n,p}(x) = \frac{\mathbb{E}_x\left[e^{\beta\ell_{n-p}}\right]}{\mathbb{E}_0\left[e^{\beta\ell_n}\right]}. \tag{4.4}$$

Moreover, according to relation (3.10), we have, for any $x \in \mathbb{Z}$,

$$U_{n,p}(x) \underset{n\to\infty}{\sim} \begin{cases} \exp\{\frac{\mathtt{F}(\beta)}{2}(|x|-p) - \beta|x| - \mathbb{1}_{\mathbb{O}}(x+p)\} & \text{if } n \in \mathbb{D} \\ \exp\{\frac{\mathtt{F}(\beta)}{2}(|x|-p) - \beta|x| + \mathbb{1}_{\mathbb{O}}(x+p)\} & \text{if } n \in \mathbb{O}, \end{cases} \tag{4.5}$$

where we used the symmetry on x. To apply the dominated convergence theorem let us note that

$$\mathbb{E}_x\left[e^{\beta\ell_{n-p}}\right] \leq \mathbb{E}_0\left[e^{\beta\ell_{n-p}}\right], \forall x \in \mathbb{Z} \quad \Rightarrow \quad U_{n,p}(x) \leq 1, \forall x \in \mathbb{Z} \quad \Rightarrow \quad U_{n,p}(b_p) \leq 1.$$

Therefore, we obtain that

$$M_p^{(\beta)} = \exp\left\{\frac{\mathtt{F}(\beta)}{2}(b_p - p) - \beta b_p + \beta\hat{\ell}_p\right\},$$

and we deduce (4.2). It is now easily checked that the process $M^{(\beta)}$ is a martingale. Indeed, setting $N_p^{(\beta)} = \ln(M_p^{(\beta)})$, and noting that $c_+(\beta) + c_-(\beta) = \beta$, we have

$$\begin{aligned} N_{p+1}^{(\beta)} &= \beta\hat{\ell}_{p+1} - c_+(\beta)\,|b_{p+1}| - c_-(\beta)\,(p+1) \\ &= \mathbb{1}_{\{b_p=0\}}[\beta(\hat{\ell}_p + 1) - \beta - c_-(\beta)\,p] + \mathbb{1}_{\{b_p\neq 0\}}[\beta(\hat{\ell}_p - c_+(\beta)\,(|b_p| + \xi_{p+1}) - c_-(\beta)\,(p+1)], \end{aligned}$$

where ξ_{p+1} is a symmetric ± 1-valued random variable independent of $\mathcal{F}_p$, representing the increment of b at time $p+1$. Hence

$$N_{p+1}^{(\beta)} = \mathbb{1}_{\{b_p=0\}} N_p^{(\beta)} + \mathbb{1}_{\{b_p\neq 0\}}[N_p^{(\beta)} - c_+(\beta)\,\xi_{p+1} - c_-(\beta)]. \tag{4.6}$$

Thus

$$\mathbb{E}_0[M_{p+1}^{(\beta)} \mid \mathcal{F}_p] = \mathbb{1}_{\{b_p=0\}} M_p^{(\beta)} + \mathbb{1}_{\{b_p\neq 0\}} M_p^{(\beta)} \cosh(c_+(\beta)) \exp(-c_-(\beta)),$$

from which the martingale property is readily obtained from the definition (4.3). □

Remark 4.2. *It should be noticed that the convergence of $\mathbb{Q}_0^{(n,\beta)}$ we have obtained on $\mathcal{F}_p$ is stronger than the weak convergence. In fact, we have been able to prove that, for any $\Gamma_p \in \mathcal{F}_p$, we have $\lim_{n\to\infty} \mathbb{Q}_0^{(n,\beta)}(\Gamma_p) = \mathbb{Q}_0^{(\beta)}(\Gamma_p)$. This property is classical in the penalization theory.*

5. The process under the new probability measure

It must be noticed that $\mathbb{Q}_0^{(\beta)}$ is a probability measure on $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geq 1})$, since $M_0^{(\beta)} = 1$. In this section we study the process $\{b_n; n \geq 1\}$ under the new probability measure $\mathbb{Q}_0^{(\beta)}$, which recovers the results of Theorem 1.1, part 2.

Proposition 5.1. *Let $\mathbb{Q}_0^{(\beta)}$ be the probability measure defined by (4.1) with $M^{(\beta)}$ given by (4.2). Then, under $\mathbb{Q}_0^{(\beta)}$:*

a) *$\{b_n; n \geq 1\}$ is a Markov process on the state space $\mathbb{Z}$ having some transition probabilities given by*

$$\mathbb{Q}_0^{(\beta)}(b_n = r \mid b_{n-1} = r-1) = \begin{cases} e^{-\beta}/2 & \text{if } r > 1 \\ 1 - e^{-\beta}/2 & \text{if } r < -1, \end{cases} \tag{5.1}$$

$$\mathbb{Q}_0^{(\beta)}(b_n = r \mid b_{n-1} = r+1) = \begin{cases} 1 - e^{-\beta}/2 & \text{if } r \geq 0 \\ e^{-\beta}/2 & \text{if } r < -1 \end{cases} \tag{5.2}$$

and

$$\mathbb{Q}_0^{(\beta)}(b_n = 1 \mid b_{n-1} = 0) = \mathbb{Q}_0^{(\beta)}(b_n = -1 \mid b_{n-1} = 0) = 1/2. \tag{5.3}$$

b) *the Laplace transform of the first return time in 0 is given by*

$$\mathbb{E}_0^{(\beta)}\left[e^{-\delta\tau_0}\right] = e^{\beta}\left(e^{\delta + \mathtt{F}(\beta)} - \left[e^{2(\delta+\mathtt{F}(\beta))} - 1\right]^{1/2}\right). \tag{5.4}$$

In particular, $\mathbb{E}_0^{(\beta)}[\tau_0] < \infty$ for any $\beta > 0$, and

$$\mathbb{E}_0^{(\beta)}[\tau_0] \sim 1 - e^{-\beta/2}, \qquad \text{when} \qquad \beta \to \infty. \tag{5.5}$$

c) *the distribution law of the excursion between two succesive zero of the process $\{b_n; n \geq 1\}$ is the same as under $\mathbb{P}_0$.*

Proof. a) Let $\Gamma_{n-2} \in \mathcal{F}_{n-2}$ arbitrary. Then

$$\mathbb{Q}_0^{(\beta)}(b_n = r \mid b_{n-1} = r-1, \Gamma_{n-2}) = \frac{\mathbb{Q}_0^{(\beta)}(b_n = r, b_{n-1} = r-1, \Gamma_{n-2})}{\mathbb{Q}_0^{(\beta)}(b_{n-1} = r-1, \Gamma_{n-2})} \tag{5.6}$$

$$= \frac{\mathbb{E}_0\left[\mathbb{1}_{\{b_n=r\}}\mathbb{1}_{\{b_{n-1}=r-1\}}\mathbb{1}_{\Gamma_{n-2}}M_n^{(\beta)}\right]}{\mathbb{E}_0\left[\mathbb{1}_{\{b_{n-1}=r-1\}}\mathbb{1}_{\Gamma_{n-2}}M_{n-1}^{(\beta)}\right]} = \frac{\mathbb{E}_0\left\{\mathbb{E}_0\left[\mathbb{1}_{\{b_n=r\}}\mathbb{1}_{\{b_{n-1}=r-1\}}\mathbb{1}_{\Gamma_{n-2}}M_n^{(\beta)} \mid \mathcal{F}_{n-1}\right]\right\}}{\mathbb{E}_0\left[\mathbb{1}_{\{b_{n-1}=r-1\}}\mathbb{1}_{\Gamma_{n-2}}M_{n-1}^{(\beta)}\right]}.$$

First, assume that $r=1$ in the latter equality Since $M_n^{(\beta)} = M_{n-1}^{(\beta)}$ if $b_{n-1}=0$, then

$$\mathbb{Q}_0^{(\beta)}(b_n = 1 \mid b_{n-1} = 0, \Gamma_{n-2}) = \frac{\mathbb{E}_0\left\{\mathbb{E}_0\left[\mathbb{1}_{\{b_n=1\}}\mathbb{1}_{\{b_{n-1}=0\}}\mathbb{1}_{\Gamma_{n-2}}M_{n-1}^{(\beta)} \mid \mathcal{F}_{n-1}\right]\right\}}{\mathbb{E}_0\left[\mathbb{1}_{\{b_{n-1}=0\}}\mathbb{1}_{\Gamma_{n-2}}M_{n-1}^{(\beta)}\right]} = \mathbb{E}_0\left[\mathbb{1}_{\{b_n=1\}} \mid \mathcal{F}_{n-1}\right] = \frac{1}{2}. \tag{5.7}$$

The same kind of computations can be performed with $\Gamma_{n-2} = \Omega$, which gives

$$\mathbb{Q}_0^{(\beta)}(b_n = 1 \mid b_{n-1} = 0, \Gamma_{n-2}) = \mathbb{Q}_0^{(\beta)}(b_n = 1 \mid b_{n-1} = 0). \tag{5.8}$$

Second, assume that $r > 1$ in (5.6). In this case, invoking (4.6) we have

$$\begin{aligned}
&\mathbb{Q}_0^{(\beta)}(b_n = r \mid b_{n-1} = r-1, \Gamma_{n-2}) \\
&= \frac{\mathbb{E}_0\left\{\mathbb{E}_0\left[\mathbb{1}_{\{b_n=r\}}\mathbb{1}_{\{b_{n-1}=r-1\}}\mathbb{1}_{\Gamma_{n-2}}M_{n-1}^{(\beta)}e^{-\xi_n c_+(\beta)-c_-(\beta)} \mid \mathcal{F}_{n-1}\right]\right\}}{\mathbb{E}_0\left[\mathbb{1}_{\{b_{n-1}=r-1\}}\mathbb{1}_{\Gamma_{n-2}}M_{n-1}^{(\beta)}\right]} \\
&= \frac{\mathbb{E}_0\left\{\mathbb{1}_{\{b_{n-1}=r-1\}}\mathbb{1}_{\Gamma_{n-2}}M_{n-1}^{(\beta)}\mathbb{E}_0\left[\mathbb{1}_{\{b_n=r\}}e^{-\xi_n c_+(\beta)-c_-(\beta)} \mid \mathcal{F}_{n-1}\right]\right\}}{\mathbb{E}_0\left[\mathbb{1}_{\{b_{n-1}=r-1\}}\mathbb{1}_{\Gamma_{n-2}}M_{n-1}^{(\beta)}\right]} \\
&= \mathbb{E}_{r-1}\left[\mathbb{1}_{\{b_1=r\}}e^{-\xi_1 c_+(\beta)-c_-(\beta)}\right] = \frac{1}{2}e^{-(c_+(\beta)+c_-(\beta))} = \frac{1}{2}e^{-\beta}.
\end{aligned} \tag{5.9}$$

Again, we can get that

$$\mathbb{Q}_0^{(\beta)}(b_n = r \mid b_{n-1} = r-1, \Gamma_{n-2}) = \mathbb{Q}_0^{(\beta)}(b_n = r \mid b_{n-1} = r-1). \tag{5.10}$$

Hence (5.8) and (5.10) prove the Markovian feature of the process $\{b_n; n \geq 1\}$ under $\mathbb{Q}_0^{(\beta)}$, while (5.7) and (5.9) prove the first equalities in (5.1) and (5.3). The other equalities can be obtained in a similar way.

b) We can write

$$\mathbb{Q}_0^{(\beta)}(\tau_0 = 2k) = \mathbb{E}_0\left[\mathbb{1}_{\{\tau_0=2k\}}M_{2k}^{(\beta)}\right] = e^{\beta-2kc_-(\beta)}\mathbb{P}_0(\tau_0 = 2k) = e^{\beta-k\mathtt{F}(\beta)}\mathbb{P}_0(\tau_0 = 2k),$$

where we used (4.2) and the fact that $2c_-(\beta) = \mathtt{F}(\beta)$. Clearly, the latter equality defines a probability measure since, thanks to (3.5),

$$\sum_{k\geq 1} e^{\beta-k\mathtt{F}(\beta)}\mathbb{P}_0(\tau_0 = 2k) = e^{\beta}\mathbb{E}_0\left[e^{-\mathtt{F}(\beta)\tau_0/2}\right] = 1.$$

Moreover, we can compute the Laplace transform of τ_0

$$(5.11)\quad \mathbb{E}_0^{(\beta)}\left[e^{-\delta\tau_0}\right] = \sum_{k\geq 1} e^{-2\delta k} e^{\beta-2kc_-(\beta)}\mathbb{P}_0(\tau_0 = 2k) = e^{\beta}\mathbb{E}_0\left[e^{-(\delta+\mathtt{F}(\beta))\tau_0}\right]$$

$$= \exp\left\{\beta - \arg\cosh\left(e^{\delta+\mathtt{F}(\beta)}\right)\right\} = \frac{e^{\beta}}{e^{\delta+\mathtt{F}(\beta)} + \left[e^{2(\delta+\mathtt{F}(\beta))}-1\right]^{1/2}}$$

$$= e^{\beta}\left\{e^{\delta+\mathtt{F}(\beta)} - \left[e^{2(\delta+\mathtt{F}(\beta))}-1\right]^{1/2}\right\}.$$

We deduce

$$(5.12)\qquad \mathbb{E}_0^{(\beta)}\left[\tau_0\right] = -\frac{d}{d\delta}\mathbb{E}_0^{(\beta)}\left[e^{-\delta\tau_0}\right]_{|\delta=0} = e^{\beta+\mathtt{F}(\beta)}\left\{\frac{1}{\left[1-e^{-2\mathtt{F}(\beta)}\right]^{1/2}} - 1\right\}.$$

By (5.12) we also get that $\lim_{\beta\to\infty}\mathbb{E}_0^{(\beta)}\left[\tau_0\right] = 1 = \lim_{\beta\to\infty} 1/\mathtt{F}'(\beta)$, by using also (3.6), while $\mathbb{E}_0^{(\beta)}\left[\tau_0\right] \neq 1/\mathtt{F}'(\beta)$. The equivalent (5.5) is also easily deduced from (5.12).

c) Thanks to the Markov property it is enough to describe the first excursion of b between 0 and τ_0. For any positive Borel function f, we have

$$\mathbb{E}_0^{(\beta)}\left[f(b_0,\ldots,b_n)\mid\tau_0=n\right] = \frac{\mathbb{E}_0\left[f(b_0,\ldots,b_n)\mathbb{1}_{\{\tau_0=n\}}M_{\tau_0}\right]}{\mathbb{E}_0\left[\mathbb{1}_{\{\tau_0=n\}}M_{\tau_0}\right]}.$$

Since, $M_{\tau_0} = e^{\beta-c_-(\beta)n}$, if $\tau_0 = n$, we obtain that

$$\mathbb{E}_0^{(\beta)}\left[f(b_0,\ldots,b_n)\mid\tau_0=n\right] = \mathbb{E}_0\left[f(b_0,\ldots,b_n)\mid\tau_0=n\right].$$

$\square$

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